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THE STABILITY OF SHELLS OF REVOLUTION WITH MICRODAMAGES†

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An approach to the study of the bifurcational stability of convex shells of revolution of double curvature is proposed, taking into account the microdamageability of an isotropic material in the form of the formation of a system of elliptic chaotically distributed microcracks throughout its volume, the concentration of which increases as the load increases. The inhomogeneous material being damaged is simulated by a continuum, the elastic symmetry and mechanism of deformation of which are associated with the character of the distribution of microstrength and the form of the interaction of the edges of the microcracks, which depends on the stressed state induced in the body. The problem of bifurcational stability in the case of shells of revolution is formulated using the concept of continued loading within the framework of the Kirchoff-Love hypothesis. As an example, problems of the stability of ellipsoidal shells in the case of an internal and external pressure are solved. © 2003 Elsevier Ltd. All rights reserved.

Because of the inhomogeneity of the strength properties of structural components made from many materials, an accumulation of microdefects in the form of plane cracks occurs as the loading level is increased [1–4]. As a consequence of such structural changes in a material, the deformation diagram is non-linear. There are two possible mechanisms of non-linear deformation. One of these is associated with an increase in the concentration of microcracks and the other is associated with the nature of the interaction of the surfaces of the microcracks (the opening or closing of a crack), which is determined by the nature of the stressed state of the body.

One of the methods of describing the combined deformation and damageability of a material, where the fractured microvolumes are simulated by spheroidal micropores, was proposed in [5, 6].

A continuum model of the deformation of elasto-brittle materials is proposed below in which the deformation is accompanied by the accumulation of damage in the form of plane microcracks which are randomly arranged throughout the volume of the body. It is assumed that these microcracks do not grow and do not interact with each other during the deformation process.

The proposed model is used to investigate the local loss of stability of convex shells of revolution. Two forms of stability loss are associated with the non-linearity of the equations of state of the material being damaged, as when investigating stability beyond the elastic limit [7, 8]. These two forms are: stability loss during continued loading (no domains of unloading) and stability loss at a constant load (the existence of domains of unloading and loading). The concentration of cracks does not change in the unloading intervals and the deformation therefore occurs linearly. During loading, the material deforms non-linearly as a consequence of the increase in the concentration of cracks, which causes a loss of resistance to deformation. Lower values of the critical loading compared with the case of a constant load are a consequence of this. For simplicity in formulating the stability problems below, the concept of continued loading is used.

1. COUPLED DEFORMATIONS AND CRACK FORMATION FOR A COMPLEX STRESSED STATE

The equations of state for a damaged material with a constant concentration of plane microdefects in the case of omnidirectional stretching, compression and a biaxial stressed state, accompanied by stretching and compression, have been obtained in [10, 11] using the energy method [9].

In general case, the relation between the macrostresses and the macrostrains for a medium simulating an isotropic material with cracks has the form

$$\varepsilon_{ij} = a_{ijkl}\sigma_{kl}, \quad i, j, k, l = 1, 2, 3 \tag{1.1}$$

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where the stresses σ_{kl} are assumed to be given in the laboratory system of coordinates $Ox_1x_2x_3$ associated with the representative volume and the strains ε_{ij} are subject to averaging. The effective compliances a_{ijkl} of the damaged material are determined by the energy method [9], which is based on the principle of the equivalence of the energy of a damaged medium and the continuum which simulates it [12]. The energy density of the strain of the model medium is written in the form

$$W = {}^1\!/_2\, a_{ijkl}\sigma_{ij}\sigma_{kl} = W^\circ + W', \quad W^\circ = {}^1\!/_2\, a^\circ_{ijkl}\sigma_{ij}\sigma_{kl} \tag{1.2}$$

where W° is the energy density of the continuum and W' is the increment in the energy density of the strain of the damaged medium associated with the release of internal energy as a consequence of the breaking of bonds accompanying the normal detachment and displacement of the crack surfaces.

The energy density released by the damaged material can be determined in the form of the work of the mutual displacement of the crack surfaces caused by the stresses which would take place in the case of a specified load in a continuum at sites which are occupied by cracks [9],

$$W' = \frac{1}{2}\sum_{k=1}^{N_0}\int_{s_k} U_i^k \sigma_{i3}^k ds_k, \quad i = 1, 2, 3 \tag{1.3}$$

which N_0 is the number of cracks per unit volume, U_i^k $(i = 1, 2, 3)$ are the displacement at the points of the surface of the kth crack, s_k is the total surface of the kth crack and σ_{i3}^k $(i = 1, 2, 3)$ are the components of the stress tensor in the natural system of coordinates of the kth crack $O^k x_1^k x_2^k x_3^k$. In the case of elliptic cracks, the axes $O^k x_i^k$ $(i = 1, 2)$ are directed along the semi-major (a) and semi-minor (b) axes of the cracks and the axis $O^k x_3^k$ is directed along the normal to their surfaces. The local stresses σ_{i3}^k, causing the displacements of the crack edges, and the average stresses σ_{ij} in the representative volume are related by the transformation

$$\sigma_{i3}^k = \sigma_{mn}\alpha_{im}^k\alpha_{3n}^k \tag{1.4}$$

where α_{im}^k, α_{3n}^k are the direction cosines of the natural system of coordinates of the kth crack, determined by the Euler angles $(\theta^k, \varphi^k, \psi^k)$, with respect to the laboratory system of coordinates.

In the case of an extensive distribution of cracks over orientations (θ, φ, ψ) and dimensions (a, b), the compliances of the damaged material are determined in terms of the characteristics of the cracks and the constants of elasticity of the continuum

$$a_{ijkl} = a^\circ_{ijkl} + a'_{ijkl} \tag{1.5}$$

Here a'_{ijkl} is the result of averaging W', that depends on distribution of the cracks with respect to their orientations and dimensions, which is completely specified by the density distribution $F(\varphi, \psi, \theta, a, b)$,

$$\langle W' \rangle = \frac{1}{8\pi^2}\int_\varphi\int_\psi\int_\theta\int_a\int_b F(\varphi, \psi, \theta, a, b) W' \sin\theta d\varphi d\psi d\theta da db$$

and, also, on the nature of the interaction of the surfaces of the microcracks, which is determined by the form of the stressed state induced in the body. In particular, the case of an isotropic material with defects in the form of plane cracks, which are distributed statistically homogeneously and isotropically throughout the volume, during omnidirectional compression the second terms on the right-hand side of equalities (1.5) will be determined by the relations [10, 11]

$$\begin{gathered} a'_{iiii} = \frac{2}{15}(1 - 3f^2)A\varepsilon, \quad a'_{iijj} = -\frac{1}{15}(1 + 2f^2)A\varepsilon \\ a'_{ijij} = \frac{2}{15}(3 - 4f^2)A\varepsilon, \quad (i, j) = 1, 2, 3; \quad A = A_1 + A_2 \end{gathered} \tag{1.6}$$

where A_1 and A_2 are quantities which determine the contribution of the crack to the increment in the energy density liberated during longitudinal and transverse shear, ε is the concentration crack and f is the coefficient of sliding friction on the crack surfaces. Formulae for the first three quantities are given below.

In the case of a complex stressed state in which stretching is combined with compression, an isotropic damaged material behaves as an anisotropic, physically non-linear medium, due to the dependence of the number of cracks which open and close during deformation on the values and signs of the tensor components of the average stresses throughout the volume. As applied to problems of the stability of shells, a stressed state of the type

$$\sigma_{11} < 0, \quad \sigma_{22} > 0, \quad \sigma_{12} \geq 0$$

is considered.

In this case, only cracks which are orientated at angles φ^k to the direction of the stretching stress open. The values of these angles satisfy the inequalities

$$(i-1)\pi - \operatorname{arctg}\lambda_1 < \varphi_k < (i-1)\pi + \operatorname{arctg}\lambda_2$$

$$\lambda_i = \sigma_{12}/|\sigma_{11}| + (-1)^i\sqrt{\sigma_{12}^2/\sigma_{11}^2 + \sigma_{22}/|\sigma_{11}|}, \quad i = 1, 2$$

In accordance with the general procedure for the method being applied, three versions of the expressions for the second terms of the compliances in relations (1.5) are obtained, depending on the nature of the interaction of the crack surfaces (henceforth, unless otherwise stated, $i, j = 1, 2$ everywhere):
(1) ideal slippage on the crack surfaces ($\sigma_{33}^k < 0, f = 0$)

$$a'_{iiii} = \left(\frac{2}{15}A + k_{iiii}^{(3)}A_3\right)\varepsilon, \quad a'_{iijj} = \left(-\frac{1}{15}A + k_{iijj}^{(3)}A_3\right)\varepsilon$$
$$a'_{ijij} = \left(\frac{2}{5}A + k_{ijij}^{(3)}A_3\right)\varepsilon \tag{1.7}$$

(2) the friction coefficient is sufficiently large so that there is no slippage of the surfaces in closed cracks ($\sigma_{33}^k < 0$, $|\sigma_{i3}^k| < f|\sigma_{33}^k|$)

$$a'_{iiii} = (k_{iiii}^{(1)}A + k_{iiii}^{(3)}A_3)\varepsilon, \quad a'_{iijj} = (k_{iijj}^{(1)}A + k_{iijj}^{(3)}A_3)\varepsilon$$
$$a'_{ijij} = (k_{ijij}^{(1)}A + k_{ijij}^{(3)}A_3)\varepsilon \tag{1.8}$$

(3) the friction forces on the crack surfaces are smaller than the local shear stresses ($\sigma_{33}^k < 0$, $|\sigma_{i3}^k| > f|\sigma_{33}^k|$)

$$a'_{iiii} = \left\{\frac{2}{15}\left[1 - f^2\left(3 - \frac{15}{2}k_{iiii}^{(3)}\right)\right]A + k_{iiii}^{(3)}A_3\right\}\varepsilon$$

$$a'_{iijj} = \left\{-\frac{1}{15}[1 + f^2(2 - 15k_{iijj}^{(3)})]A + k_{iijj}^{(3)}A_3\right\}\varepsilon \tag{1.9}$$

$$a'_{ijij} = \left\{\frac{2}{15}\left[3 - f^2\left(4 - \frac{15}{2}k_{ijij}^{(3)}\right)\right]A + k_{ijij}^{(3)}A_3\right\}\varepsilon$$

Here

$$k_{iiii}^{(m)} = \frac{2m}{15\pi}\alpha + (-1)^i\frac{7m-5}{60\pi}S_2 + \frac{3m-5}{120\pi}S_4$$

$$k_{1122}^{(m)} = \frac{3m-5}{30\pi}\left(\alpha - \frac{1}{4}S_4\right), \quad k_{1212}^{(m)} = \frac{m+5}{15\pi}\alpha - \frac{3m-5}{30\pi}S_4; \quad m = 1, 3$$

$$\alpha = \alpha_1 + \alpha_2, \quad \alpha_i = \operatorname{arctg}\lambda_i, \quad S_n = \sin n\alpha_1 + \sin n\alpha_2; \quad n = 2, 4$$

$$A = A_1 + A_2, \quad A_i = \frac{1-\nu}{E_0}R_i(k, \nu_0), \quad A_3 = \frac{1-\nu_0^2}{E_0\mathbf{E}(k)}$$

$$R_i(k, \nu_0) = k^2[(k^2 - \nu_0(1-(i-1)(1-k_1^2)))\mathbf{E}(k) + (-1)^{i+1}\nu_0 k_1^2\mathbf{K}(k)]^{-1}$$

$$k^2 = 1 - b^2/a^2, \quad k_1^2 = 1 - k^2$$

$\mathbf{K}(k)$, $\mathbf{E}(k)$ are complete elliptic integrals of the second kind, and $\varepsilon = {}^4\!/_3\pi N_0\langle ab^2\rangle$ is a small parameter which determines the concentration of the cracks [9].

In the case of circular cracks of radius a, the following relations hold

$$A_1 = A_2 = \frac{4(1-\nu_0^2)}{\pi(2-\nu_0)E_0}, \quad A_3 = \frac{2(1-\nu_0^2)}{\pi E_0}, \quad \varepsilon = \frac{4\pi}{3}N_0\langle a^3\rangle$$

In the general case, the technical constants of the damaged material are determined in terms of the compliance by the relations

$$\frac{1}{E_{ii}} = \frac{1}{E_0} + a'_{iiii}, \quad \frac{1}{G_{ij}} = \frac{1}{G_0} + a'_{ijij}, \quad \frac{\nu_{ij}}{E_{ii}} = \frac{\nu_0}{E_0} - a'_{jjii} \tag{1.10}$$

Here, E_{ii}, G_{ij}, ν_{ij} are the normal moduli of elasticity, the shear modulus and Poisson's ratio.

In using the relations presented above to describe the process of complex strain of elasto-brittle materials and the formation of cracks in them, it is necessary to find how the bulk concentration of microcracks ε depends on the loading parameters. Daniels' structural model for the accumulation of defects [13] is suitable for this purpose. The change in the bulk concentrations of microcracks ε depends on the mechanism of microfracture in the material, the distribution of strength properties throughout the volume and, also, on the loading history.

Microfracture of the separation type is considered below as an example. Fracture associated with shear can also be treated similarly. The relations of the first theory of strength [1–4]

$$\sigma_n \geq \sigma \tag{1.11}$$

are taken as the criterion of failure of the structural elements, where σ is a random quantity which can stand for the limiting values of the stretching or compression stresses causing the failure of the structural elements of the material. It is assumed that, when the stretching stress σ_n reaches the value σ, a microcrack is formed in the corresponding area, the plane of which is normal to the direction of action of the stress σ_n. In the case of a compression stress σ_n, the microcracks are for the most part orientated parallel to the stress σ_n [2, 3].

If a sphere of a certain radius is chosen as the representative volume in which the average stresses σ_{ij} $(i, j = 1, 2, 3)$ are given, the normal stress σ_n on a small area, the orientation of the normal to which is given by the spherical coordinates θ (latitude) and φ (longitude), will be given by the expression

$$\begin{aligned}\sigma_n = {} & \sigma_{11}\cos^2\varphi\sin^2\theta + \sigma_{22}\sin^2\varphi\sin^2\theta + \sigma_{33}\cos^2\theta + \\ & + 2\sigma_{12}\sin\varphi\cos\varphi\sin\theta + 2\sigma_{13}\cos\varphi\sin\theta\cos\theta + 2\sigma_{23}\sin\varphi\sin\theta\cos\theta\end{aligned} \tag{1.12}$$

The true stretching stress σ_n' on this small area as a consequence of the reduction in the supporting area of the section is represented by the relation

$$\sigma_n' = \sigma_n/[1 - P_n(\sigma_n')]$$

where $P_n(\sigma_n')$ is a specific portion of the area of intersection of the failed structural elements. The concentration of plane microdefects in a random cross-section of the representative volume is therefore defined by the probability $P_n(\sigma_n' \geq \sigma)$ that the normal stress values σ_n' will not be less than the ultimate strength of the particles of the microstructure σ, which is a random quantity. The supporting area does not change $(\sigma_n' = \sigma_n)$ in the case of compression $(\sigma_n < 0)$. By analogy with the well-known approach [13], the exponential law

$$P(\sigma) = \begin{cases} 0, & \sigma < \sigma_0 \\ (\sigma - \sigma_0)^{\eta}/(\sigma_c - \sigma_0)^{\eta}, & \sigma_0 \le \sigma \le \sigma_c \\ 1, & \sigma > \sigma_c \end{cases} \tag{1.13}$$

is used to approximate the distribution of the mechanical properties of the crystallites and grains of different orientation.

The distribution parameters η, σ_0, σ_c are found using selected values by the method of moments, for example [14]. The principal moments, that is, the average microstrength $\langle\sigma\rangle$ and the dispersion D^2 for distribution (1.13), have the form

$$\langle\sigma\rangle = \frac{\eta}{\eta+1}(\sigma_c - \sigma_0) + \sigma_0, \quad D^2 = \frac{\eta}{\eta+2}\sigma_c^2 - \frac{2\eta}{\eta+1}\langle\sigma\rangle\sigma_c + \langle\sigma\rangle^2 \tag{1.14}$$

When (1.13) is taken into account, the mean probability of the failure of structural components intersecting a unit area of the surface of the representative volume will be given by the relation

$$p = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} P_n \sin\theta d\theta d\varphi, \quad P_n = P(\sigma_n') \tag{1.15}$$

The physical meaning of the quantity p consists of the fact that it represents the specific fraction of a unit area of the surface of a sphere on which the normal stresses σ_n exceed the ultimate strength σ of the microparticles which intersect the surface of the sphere. At the same time, in the case of stretching, the particles crack along surfaces which are normal to σ_n while, in the case of compression, they crack in the direction of the action of σ_n. The bulk concentration of plane microdefects ε, which appears in relations (1.6)–(1.9), will be given by the ratio of the number of failed microparticles N_p to their bulk number N in the representative volume. By using the technique employed in petrography to analyse thin slices of deposits [15], it can be shown that $\varepsilon = p$.

Hence, the coupled process of deformation and dispersed fracture in the form of the formation of a system of stochastically orientated plane microcracks is simulated by the closed system of non-linear equations (1.1), (1.6)–(1.9), (1.11)–(1.13) and (1.15).

As applied to the formulation and solution of problems of the stability of shells taking the microdamageability of the material into account, the governing equations presented above are later required in the form

$$\sigma_{11} = \frac{E_{11}}{1-\nu_{12}\nu_{21}}(\varepsilon_{11} + \nu_{21}\varepsilon_{22}), \quad \sigma_{22} = \frac{E_{22}}{1-\nu_{12}\nu_{21}}(\varepsilon_{22} + \nu_{12}\varepsilon_{11}), \quad \sigma_{12} = G_{12}\varepsilon_{12}$$
$$\varepsilon_{11} = \frac{1}{E_{11}}(\sigma_{11} - \nu_{12}\sigma_{22}), \quad \varepsilon_{22} = \frac{1}{E_{22}}(\sigma_{22} - \nu_{21}\sigma_{11}), \quad \varepsilon_{12} = \frac{1}{G_{12}}\sigma_{12} \tag{1.16}$$

Here, E_{ii}, G_{12}, ν_{ij} are the elasticity characteristics, defined by formulae (1.10) depending on the stressed state which is induced in the body.

2. FORMULATION OF THE PROBLEM OF THE STABILITY OF SHELLS OF REVOLUTION MADE OF DAMAGED MATERIAL

The local loss of stability of thin shells of revolution in the elastic domain of deformation has been considered earlier [16]. A similar problem has been solved in [8] beyond the elastic limit. Below, we propose a method for solving problems on the local loss of stability of closed shells of double curvature which is accompanied by the formation of plane microdefects in the material. A shell of thickness h is referred to the system of coordinates $O_m x_1 x_2 x_3$ connected with the middle surface. The coordinates x_1, x_2, x_3 are read off in the meridional direction, the circumferential direction and the direction perpendicular to the middle surface respectively. Displacements of the points of the middle surface in the above-mentioned directions are denoted by the letters u, υ, w. In the case of the type of shells being considered, the apparatus of the theory of thin shells [7, 8, 16] can be used to solve stability problems. Then, within the framework of the Kirchhoff–Love hypothesis, the deformations at an arbitrary point of the shell will be given by the relations

$$\varepsilon_{ij} = e_{ij} + x_3\chi_{ij} \tag{2.1}$$

where the strains, curvatures and torsion of the middle surface have the form

$$\begin{aligned} e_{11} &= u_{,1} - k_1 w, \quad e_{22} = v_{,2} - k_2 w, \quad e_{12} = u_{,2} + v_{,1} \\ \chi_{11} &= -w_{,11}, \quad \chi_{22} = -w_{,22}, \quad \chi_{12} = -2w_{,12} \end{aligned} \tag{2.2}$$

The form of the equations of the local loss of stability of shells in a combined form is independent of the properties of the material [7, 8, 16]

$$\begin{aligned} &M_{11,11} + 2M_{12,12} + M_{22,22} + h(k_1\Phi_{,22} + k_2\Phi_{,11}) + h(\sigma_{11}^\circ w_{,11} + 2\sigma_{12}^\circ w_{,12} + \sigma_{22}^\circ w_{,22}) = 0 \\ &\bar{e}_{11,22} + \bar{e}_{22,11} - \bar{e}_{12,12} = -k_1 w_{,22} - k_2 w_{,11} \end{aligned} \tag{2.3}$$

Here,

$$M_{ij} = \int_{-h/2}^{h/2} x_3\bar{\sigma}_{ij}dx_3$$

$\bar{\sigma}_{ij}$, $\bar{e}_{ij}$, χ_{ij}, w are the increments of the moments and stresses in the shell as a consequence of flexure and, also, of the strains, curvatures and deflections of the middle surface in the perturbed state, σ_{ij}° are the stresses in the basic momentless stressed state and k_1 and k_2 are the principal curvatures of the shell in the meridional and circumferential directions.

To these equations it is necessary to add the expressions for the perturbations for the membrane stresses in terms of the stress function Φ

$$\bar{\bar{\sigma}}_{11} = \Phi_{,22}, \quad \bar{\bar{\sigma}}_{22} = \Phi_{,11}, \quad \bar{\bar{\sigma}}_{12} = -\Phi_{,12} \tag{2.4}$$

The increments in the stresses $\bar{\sigma}_{ij} = \sigma_{ij} - \sigma_{ij}^\circ$ and the strains $\bar{\varepsilon}_{ij} = \varepsilon_{ij} - \varepsilon_{ij}^\circ$ are determined by the variation of Eqs (1.16) in the neighbourhood of the basic stressed state. These equations connect the final values of the stresses and strains in the case of a medium which is damaged, taking into account the dependence of the moduli of elasticity on the concentration of microcracks ε. As a result of variation in the neighbourhood of the basic stress-strain state, the perturbations of the stresses and strains are represented in the form

$$\bar{\sigma}_{ii} = a_{i1}\bar{\varepsilon}_{11} + a_{i2}\bar{\varepsilon}_{22} + a_{i3}\bar{\varepsilon}_{12}, \quad \bar{\sigma}_{12} = a_{31}\bar{\varepsilon}_{11} + a_{32}\bar{\varepsilon}_{22} + a_{33}\bar{\varepsilon}_{12} \tag{2.5}$$

$$\bar{\varepsilon}_{ii} = A_{i1}\bar{\sigma}_{11} + A_{i2}\bar{\sigma}_{22} + A_{i3}\bar{\sigma}_{12}, \quad \bar{\varepsilon}_{12} = A_{31}\bar{\sigma}_{11} + aA_{32}\bar{\sigma}_{22} + A_{33}\bar{\sigma}_{12} \tag{2.6}$$

The coefficients a_{ij}, A_{ij}, which are given by the relations

$$a_{11} = \frac{\partial\sigma_{11}}{\partial\varepsilon_{11}}, a_{12} = \frac{\partial\sigma_{11}}{\partial\varepsilon_{22}}, \ldots, A_{11} = \frac{\partial\varepsilon_{11}}{\partial\sigma_{11}}, A_{12} = \frac{\partial\varepsilon_{11}}{\partial\sigma_{22}}, \ldots$$

have the form

$$\begin{aligned} a_{ii} &= \frac{E_{ii}}{1-\nu_{12}\nu_{21}} - \alpha_{ii}\frac{\partial\varepsilon}{\partial\varepsilon_{ii}}, \quad a_{12} = \frac{\nu_{21}E_{11}}{1-\nu_{12}\nu_{21}} - \alpha_{11}\frac{\partial\varepsilon}{\partial\varepsilon_{22}} \\ a_{21} &= \frac{\nu_{12}E_{22}}{1-\nu_{12}\nu_{21}} - \alpha_{22}\frac{\partial\varepsilon}{\partial\varepsilon_{11}}, \quad \alpha_{3i} = -\alpha_{12}\frac{\partial\varepsilon}{\partial\varepsilon_{ii}}, \quad a_{33} = G_{12} - \alpha_{12}\frac{\partial\varepsilon}{\partial\varepsilon_{12}} \\ a_{i3} &= -\alpha_{ii}\frac{\partial\varepsilon}{\partial\varepsilon_{12}}, \quad \alpha_{ii} = \sigma_{ii}^\circ E_{ii}\frac{\alpha'_{iiii}}{\varepsilon}, \quad \alpha_{12} = \sigma_{12}^\circ G_{12}\frac{\alpha'_{1212}}{\varepsilon} \\ A_{ii} &= \frac{1}{E_{ii}} + \beta_{ii}\frac{\partial\varepsilon}{\partial\sigma_{ii}}, \quad A_{12} = -\frac{\nu_{12}}{E_{11}} + \beta_{11}\frac{\partial\varepsilon}{\partial\sigma_{22}}, \quad A_{i3} = \beta_{ii}\frac{\partial\varepsilon}{\partial\sigma_{12}} \\ A_{21} &= -\frac{\nu_{21}}{E_{22}} + \beta_{22}\frac{\partial\varepsilon}{\partial\sigma_{11}}, \quad A_{3i} = \beta_{12}\frac{\partial\varepsilon}{\partial\sigma_{ii}}, \quad A_{33} = \frac{1}{G_{12}} + \beta_{12}\frac{\partial\varepsilon}{\partial\sigma_{12}} \end{aligned} \tag{2.7}$$

$$\beta_{11} = (\sigma^{\circ}_{11} - \nu_{12}\sigma^{\circ}_{22})\frac{a'_{1111}}{\varepsilon}, \quad \beta_{22} = (\sigma^{\circ}_{22} - \nu_{21}\sigma^{\circ}_{11})\frac{a'_{2222}}{\varepsilon}$$

$$\beta_{12} = \sigma^{\circ}_{12}\frac{a'_{1212}}{\varepsilon}$$

The relations presented above hold for the general case of the basic stressed state of shells of diverse geometry. A problem on the local loss of stability of convex shells of revolution in the case of a uniform stressed state is considered next. In this case, Eqs (2.3), when relations (2.4)–(2.7), which hold for membrane and flexural stresses and strains, are taken into account, take the form

$$\begin{gathered} D[a_1 w_{,1111} + a_2 w_{,1122} + a_3 w_{,2222} + 2a_4 w_{,1112} + 2a_5 w_{,1222}] - \\ - T^{\circ}_{11} w_{,11} - T^{\circ}_{22} w_{,22} - 2T^{\circ}_{12} w_{,12} - h(k_1\Phi_{,22} + k_2\Phi_{,11}) = 0 \\ \bar{A}_1\Phi_{,1111} + \bar{A}_2\Phi_{,1122} + \bar{A}_3\Phi_{,2222} - \bar{A}_4\Phi_{,1112} - \bar{A}_5\Phi_{,1222} = E_0 h(k_1 w_{,22} + k_2 w_{,11}) \end{gathered} \tag{2.8}$$

where

$$\bar{a}_{ij} = a_{ij}/E_0, \quad \bar{A}_{ij} = E_0 A_{ij}$$

$$a_1 = \bar{a}_{11}, \quad a_2 = \bar{a}_{12} + \bar{a}_{21} + 4\bar{a}_{33}, \quad a_3 = \bar{a}_{22}$$

$$a_4 = \bar{a}_{13} + \bar{a}_{31}, \quad a_5 = \bar{a}_{23} + \bar{a}_{32}$$

$$\bar{A}_1 = \bar{A}_{22}, \quad \bar{A}_2 = \bar{A}_{12} + \bar{A}_{21} + \bar{A}_{33}, \quad \bar{A}_3 = \bar{A}_{11}$$

$$\bar{A}_4 = \bar{A}_{32} + \bar{A}_{23}, \quad \bar{A}_5 = \bar{A}_{13} + \bar{A}_{31}$$

$$D = E_0 h^3/12, \quad T^{\circ}_{ij} = \sigma^{\circ}_{ij} h$$

(T°_{ij} are the linear shear stresses of the subcritical stressed state).

3. THE AXISYMMETRIC LOADING OF SHELLS

The local loss of stability of a closed shell of revolution under the action of an internal or external uniform pressure of intensity q is considered as an illustrative example. In this case, the stresses in the basic stressed state are given by the relations

$$T^{\circ}_{11} = h\sigma^{\circ}_{11} = \pm\frac{qR_2}{2}, \quad T^{\circ}_{22} = h\sigma^{\circ}_{22} = \pm\frac{qR_2}{2}(2 - \chi), \quad \chi = \frac{k_1}{k_2} \tag{3.1}$$

The upper sign refers to the case of an internal pressure.

The solution of the system of equations (2.8) is represented in the form [8, 16]

$$w = A\sin(k_2\lambda x_1)\sin(k_2 n x_2), \quad \lambda = \chi m \tag{3.2}$$

where m and n are wave numbers in the meridional and circumferential directions.

The critical pressure is given by the formula

$$q_* = \frac{2k_2}{1 \mp (2-\chi)\gamma}\left[Dk_2^2\lambda^2(a_1 + a_2\gamma + a_3\gamma^2)^2 + \frac{E_0 h(1+\gamma\chi)^2}{\lambda^2(\bar{A}_1 + \bar{A}_2\gamma + \bar{A}_3\gamma^2)}\right] \tag{3.3}$$

The upper and lower signs refer to the cases of internal and external pressures, respectively. When $\gamma = n^2/\lambda^2 \gg 1$, the minimum value of the critical pressure is given by the formula

$$q_* = \frac{2E_0 h^2 k_1 k_2^2}{\mp\sqrt{3}(2k_2 - k_1)}\sqrt{\frac{a_3}{\bar{A}_3}} \tag{3.4}$$

In the case of a continuous elastic material ($\varepsilon = 0$), expression (3.4) reduces to the well-known formula [16]

$$q_* = \frac{2E_0}{\sqrt{3(1-\nu_0^2)}} \frac{h^2 k_1 k_2^2}{2k_2 - k_1} \tag{3.5}$$

In formulae (3.4) and (3.5), k_1, k_2 are certain mean values of the principal curvatures of segments of the shell, which are bounded by the nodal lines of the local forms of loss of stability.

Of the countless set of values of q_* given by formulae (3.4) and (3.5), the minimum value is the required value. In the case of closed convex shells, domains of the surface of a shell which contain tangents to the generatrix which are parallel or perpendicular to the axis of revolution will be the weakest segments in the sense of their local stability. In particular, in elliptic shells these domains are located on the equator and at the poles. The curvatures at the poles (the subscript p) and the equatorial points (the subscript e) in a shell with semi-axes $\bar{a}$ (the radius of the equator) and $\bar{b}$ are given by the expressions

$$k_{1p} = \bar{b}/\bar{a}^2, \quad k_{2p} = \bar{b}/\bar{a}^2; \quad k_{1e} = \bar{a}/\bar{b}^2, \quad k_{2e} = 1/\bar{a} \tag{3.6}$$

It follows from formula (3.4) when expressions (3.6) are taken into account that the critical pressure values for ellipsoidal shells, in the case of an external action which depends on the ratio of the semi-axes, will be given by the formulae

$$q_* = q_*^\circ \times \begin{cases} (2\bar{b}^2/\bar{a}^2 - 1)^{-1}, & \bar{a} < \bar{b} \\ \bar{b}^2/\bar{a}^2, & \bar{a} > \bar{b} \end{cases}, \quad q_*^\circ = \frac{2h^2 E_0}{\sqrt{3}\bar{a}^2}\sqrt{\frac{a_3}{\bar{A}_3}} \tag{3.7}$$

Local stability loss close to the equator is possible in the case of an internal pressure

$$q_* = q_*^\circ (1 - 2\bar{b}^2/\bar{a}^2)^{-1}, \quad \bar{a} > \bar{b} \tag{3.8}$$

The diversity of possible versions of the crack formation in shells is limited by a consideration of microdefects in the form of circular cracks and by the choice of a material with the following characteristics

$$E = 4.2 \times 10^{11} \text{ Pa}, \quad \upsilon_0 = 0.2, \quad \langle\sigma\rangle = 1.9 \times 10^9 \text{ Pa}$$
$$D = 0.672 \times 10^9 \text{ Pa}, \quad f = 0 \tag{3.9}$$

In the case of a two-parameter distribution of microstrength (formula (1.13) when $\sigma_0 = 0$), it follows from expression (1.14) in the case of a material with the parameters (3.9) that

$$\eta = 2, \quad \sigma_c = 2.8 \times 10^9 \text{ Pa} \tag{3.10}$$

The concentration of cracks when relations (1.12), (1.13), (1.15) and (3.10) are taken into account will be given by the formula

$$\varepsilon = \frac{1}{15\sigma_c^2}[(3\sigma_{11}^{\circ 2} + 2\sigma_{11}^\circ \sigma_{22}^\circ + 2\sigma_{22}^{\circ 2})] \tag{3.11}$$

The expressions for the stresses in the subscritical state in the case of external pressure (a) and internal pressure (b) have the form

$$\text{(a) } \sigma_{11}^\circ = \sigma_{22}^\circ = -\frac{q_* \bar{a}^2}{2h\bar{b}}, \quad \text{(b) } \sigma_{11}^\circ = \frac{q_* \bar{a}}{2h}, \quad \sigma_{22}^\circ = \frac{q_* \bar{a}}{2h}\left(2 - \frac{\bar{a}^2}{\bar{b}^2}\right) \tag{3.12}$$

It is seen from the last formula (3.12) that stability loss in the case of internal pressure is possible when $\bar{a}^2/\bar{b}^2 > 2$. The coefficients $a_3, \bar{A}_3$, in accordance with expressions (2.7), can be represented in the form

$$a_3 = \frac{E_{22}}{E_0(1-\nu_{12}\nu_{21})}\left[1+\left(\alpha_{11}-\nu_{12}\alpha_{22}\frac{\partial\varepsilon}{\partial\sigma_{11}^\circ}\right)\right]\left(1+\alpha_{11}\frac{\partial\varepsilon}{\partial\sigma_{11}^\circ}+\alpha_{22}\frac{\partial\varepsilon}{\partial\sigma_{22}^\circ}\right)^{-1}$$

$$\bar{A}_3 = \frac{E_0}{E_{11}} + \frac{8E_0\sigma_{11}^\circ(\sigma_{11}^\circ-\nu_{12}\sigma_{22}^\circ)a'_{1111}}{15\sigma_c^2}\frac{}{\varepsilon} \tag{3.13}$$

In the case of external pressure, by virtue of the simulation of the damaged material as an isotropic medium, the constants of elasticity have the form

$$E_{11} = E_{22} = \frac{E_0}{1+E_0a'_{1111}}, \quad \nu_{12} = \nu_{21} = E_{11}\left(\frac{\nu_0}{E_0}-a'_{1122}\right)$$

where a'_{1111} and a'_{1122} are given by expressions (1.6). In the case of an internal pressure with relations (3.13), the quantities E_{22}, E_{11}, ν_{21}, ν_{12}, a'_{2222} and a'_{1111}, which are given by formulae (1.7) and (1.0) when $\lambda_1 = \lambda_2 = (\bar{a}^2/\bar{b}^2-2)^{-1/2}$, correspond to the parameters E_{11}, E_{22}, ν_{12}, ν_{21}, a'_{1111} and a'_{2222}.

Expressions (3.7) and (3.8) are complex functions of q_*, since the coefficients a_3 and A_3 are associated with the load. Direct determination of q_* is therefore difficult. However, it is not essential when estimating the effect of the damageability of shells on their stability. A sequence of values of q_* can be specified and the corresponding values of the relative thickness of the shells $h/\bar{a}$ can be found using formulae (3.7) and (3.8). The results to such calculations for oblate ellipsoidal shells ($\bar{a} = 2\bar{b}$) in the case of internal pressure, taking into account (subscript +) and ignoring (subscript –) the vulnerability to damage, are presented below

$q_* \times 10^{-2}$	792	3175	7163	12782	20068	29066	52415	83342
$\varepsilon \times 10^5$	183	733	1650	2933	4583	6600	11733	18333
$(h/\bar{a})_+ \times 10^6$	283	567	853	1141	1434	1730	2340	2976
$(h/\bar{a})_- \times 10^6$	282	565	848	1131	1414	1607	2262	2826

As might have been expected, the effect of the vulnerability to damage of the material on its stability increases as the relative thickness of the shells increases.

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